PERFECT BUT NOT GENERATING DELAUNAY POLYTOPES

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ABSTRACT. In his seminal 1951 paper "Extreme forms" Coxeter [Co51] observed that for $n \geq 9$ one can add vectors to the perfect lattice A_9 so that the resulting perfect lattice, called A_9^2 by Coxeter, has exactly the same set of minimal vectors. An inhomogeneous analog of the notion of perfect lattice is that of a lattice with a perfect Delaunay polytope: the vertices of a perfect Delaunay polytope are the analogs of minimal vectors in a perfect lattice. We find a new infinite series $P(n, s)$ for $s \geq 2$ and $n + 1 \geq 4s$ of n-dimensional perfect Delaunay polytopes. A remarkable property of this series is that for certain values of s and all $n \geq 13$ one can add points to the integer affine span of $P(n, s)$ in such a way that $P(n, s)$ remains a perfect Delaunay polytope in the new lattice. Thus, we have constructed an inhomogeneous analog of the remarkable relationship between A_9 and A_9^2.

1. INTRODUCTION

Given a n-dimensional lattice L, a polytope D is called a *Delaunay polytope* if the set of its vertices is $S \cap L$ with S being a sphere containing no lattice points in its interior. If $(v_1, \ldots, v_n)$ is a basis of L then the Gram matrix $Q = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n}$ characterizes L up to isometry. It has long been observed that for computations it is preferable to work with Gram matrices instead of lattices. Then one defines $S_{>0}^n$ the cone of positive definite $n \times n$-symmetric matrices, identifies the quadratic forms with symmetric matrices and defines $A[X] = X^t A X$ for a column vector X and a symmetric matrix A.

Voronoi [Vo08] remarked that if D is a polytope with coordinates in $\mathbb{Z}^n$ then the condition that D is a Delaunay polytope is expressed by linear equalities and inequalities on the coefficients of the Gram matrix.

That is if one defines

$$\mathrm{SC}(D) = \left\{ \begin{array}{ll} Q \in S^n_{>0} & : \quad \exists c \in \mathbb{R}^n, r > 0 \text{ such that} \\ & Q[v - c] = r \text{ for } v \in \mathrm{vert}\, D \\ & \text{and } Q[v - c] > r \text{ for } v \in \mathbb{Z}^n - \mathrm{vert}\, D \end{array} \right\}$$

then $\mathrm{SC}(D)$ (called *Baranovskii cone* in [Sc09]) is a polyhedral cone. The dimension of $\mathrm{SC}(D)$ is called the *rank* of D. D is called *perfect* if it is of rank 1 (see [Er92] and [DGL93] for more details).

The only perfect Delaunay polytope of dimension $n \leq 6$ are the interval $[0, 1]$ and Schläfli polytope 2_{21}, which are Delaunay polytopes of the root lattices A_1 and E_6 (see [DD04]). Several infinite series of perfect Delaunay polytopes were built in [Er02], [Du05] and [Gr06]. Some, conjectured to be complete, lists are given in [DER07] for dimension 7 and 8. A polytope P is called *centrally symmetric* if there exist a point c, called *center*, such that for any vertex $v \in P$ we have $2c - v \in P$. In this paper for every $4s \leq n + 1$, we build a Delaunay polytope $P(n, s)$ such that:

(i) $P(n, s)$ has dimension n, is centrally symmetric and has $2\binom{n+1}{s}$ vertices.

(ii) $P(n, s)$ is perfect for $s \geq 2$.

Given a Delaunay polytope P in a lattice L, we denote by $L(P)$ the set of lattice points that can be expressed as integral sum of vertices of P. P is *generating* if $L(P)$ coincides with L.

All perfect Delaunay polytopes known so far were generating and the main interest of $P(n, s)$ is that if

$$6s < \left\{ \begin{array}{ll} n + 1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even,} \end{array} \right.$$

then there exists a lattice L' such that $P(n, s)$ is a Delaunay polytope in L' and $L(P) \neq L'$.

The polytope $P(7, 2)$ is the Gosset polytope 3_{21}, which is a Delaunay polytope of the root lattice E_7 and $P(8, 2)$ is the Delaunay polytope D_2^8 of [DER07]. The infinite series $P(n, s)$ were found by looking at D_2^8 and the lattice L' was found by an exhaustive search using the computer package [Du08].

2. THE LATTICE A_n

The lattice A_n is defined as

$$\mathsf{A}_n = \left\{ x = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \quad : \quad \sum_{i=0}^{n} x_i = 0 \right\}.$$

A_n is an n-dimensional lattice, but best seen as embedded into $\mathbb{R}^{n+1}$ with the standard Euclidean metric $\sum_{i=0}^{n} x_i^2$. Define $(e_i)_{1 \leq i \leq n+1}$ the standard basis of $\mathbb{R}^{n+1}$.

It is often useful to think of A_n as a point lattice. More formally, define

$$V_{n,s} = \left\{ x = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \quad : \quad \sum_{i=0}^{n} x_i = s \right\}.$$

Then the difference set $V_{n,s} - V_{n,s}$ is the lattice A_n. Let

$$J(n+1, s) = \operatorname{conv} \left\{ x \in \{0,1\}^{n+1} \quad : \quad \sum_{i=0}^{n} x_i = s \right\}.$$

It is easily seen that $J(n+1, s)$ is a lattice polytope in the point lattice $V_{n,s}$. Since $V_{n,s} - V_{n,s} = \mathsf{A}_n$, we know that A_n contains lattice polytopes isometric to $J(n+1, s)$.

For $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$, we define

$$q_{\alpha_0, \ldots, \alpha_n}(x) = \sum_{i=0}^{n} \alpha_i x_i^2$$

and denote by $\mathcal{Q}P$ the cone of all $q_{\alpha_0, \ldots, \alpha_n}$ with $\alpha_i > 0$. Clearly the polytopes $J(n+1, s)$ are Delaunay polytopes of A_n for the scalar product induced by $q \in \mathcal{Q}P$.

The following theorem is a reformulation of Proposition 8 of [BaGr01].

Theorem 1. *(i) The lattice A_n has n translation classes of Delaunay polytopes. These classes are represented by polytopes $J(n+1, s)$ for $1 \leq s \leq n$.*

(ii) The scalar products on A_n having the polytopes $J(n+1, s)$ as Delaunay polytopes are the ones induced by some $q \in \mathcal{Q}P$.

According to the terminology of [BaGr01] this means that the non-rigidity degree of A_n is $n + 1$. Note that the forms $x_0^2, \ldots, x_n^2$ remain independent when restricted to A_n. One classic example is the Delaunay tessellation of A_3: It is formed by the regular simplex $J(4,1)$, its antipodal $J(4,3)$ and the regular octahedron $J(4,2)$.

Clearly, the rank of the polytopes $J(n+1, 1)$ and $J(n+1, n)$ is $\frac{n(n+1)}{2}$ since those polytopes are n-dimensional simplices.

Theorem 2. *Let $n, s \in \mathbb{N}$ and $2 \leq s \leq n - 1$.*

(i) The rank of $J(n+1, s)$ is $n + 1$ and every scalar product on A_n having $J(n+1, s)$ as Delaunay is induced by some $q \in \mathcal{Q}P$.

(ii) The center $c_{\alpha_0,\ldots,\alpha_n}$ of the empty ellipsoid around $J(n+1,s)$ with respect to the quadratic form $q_{\alpha_0,\ldots,\alpha_n}$ is given by

$$c_{\alpha_0,\ldots,\alpha_n} = \left(\frac{1}{2} + \frac{C}{\alpha_0}, \ldots, \frac{1}{2} + \frac{C}{\alpha_n}\right) \text{ with } C = \frac{s - \frac{n+1}{2}}{\sum_{i=0}^{n} \frac{1}{\alpha_i}}.$$

Proof. For $i = 1, \ldots, n$ define $v_i = e_i - e_0$. The norm of a vector $x = \sum_{i=0}^{n} x_i e_i \in \mathsf{A}_n$ with respect to $q_{\alpha_0,\ldots,\alpha_n}$ is

$$
\begin{aligned}
q_{\alpha_0,\ldots,\alpha_n}(x) &= q_{\alpha_0,\ldots,\alpha_n}(-(\textstyle\sum_{i=1}^{n} x_i)e_0 + \sum_{i=1}^{n} x_i e_i) \\
&= \alpha_0(\textstyle\sum_{i=1}^{n} x_i)^2 + \sum_{i=1}^{n} \alpha_i x_i^2 \\
&= X^t A_{\alpha_0,\ldots,\alpha_n} X
\end{aligned}
$$

where $X = (x_1, \ldots, x_n)^t$, and

$$A_{\alpha_0,\ldots,\alpha_n} = \begin{pmatrix} \alpha_0 + \alpha_1 & \alpha_0 & \cdots & \alpha_0 \\ \alpha_0 & \alpha_0 + \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_0 \\ \alpha_0 & \cdots & \alpha_0 & \alpha_0 + \alpha_n \end{pmatrix}.$$

Expressed in terms of the basis $(v_i)_{1 \le i \le n}$ the polytope $J(n+1,s)$ is written as

$$J'(n+1,s) = \text{conv}\left\{(x_1, \ldots, x_n) \in \{0,1\}^n \text{ with } s - \sum_{i=1}^{n} x_i \in \{0,1\}\right\}.$$

Theorem 1, (ii) then implies that if $\alpha_i > 0$, then $A_{\alpha_0,\ldots,\alpha_n} \in \text{SC}(J'(n+1,s))$. Let us now take $A = (a_{i,j})_{1 \le i,j \le n} \in \text{SC}(J'(n+1,s))$.

Select a three element subset $S = \{s_1, s_2, s_3\}$ of $\{1, \ldots, n\}$ and a vector $v \in \{0,1\}^n$. Consider the polytope

$$J_{S,v} = \text{conv}\{w \in \text{vert } J'(n+1,s) \quad : \quad w_i = v_i \text{ for } i \notin S\}.$$

If one chooses v such that $\sum_{i \notin S} v_i = s-2$, then $J_{S,v}$ is affinely equivalent to the octahedron $J(4,2)$. The quadratic form $q(x) = X^t A X$ induces a quadratic form q_S on the affine space spanned by $J_{S,v}$ with $q_S(Y) = Y^t A_S Y$, $Y = (x_{s_1}, x_{s_2}, x_{s_3})^t$ and $A_S = (a_{i,j})_{i,j \in S}$.

The rank of the octahedron $J(4,2)$ is equal to 4 as proved on page 232 of [DeLa97]. The quadratic form $A_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ with $\alpha_i > 0$ has 4 independent coefficients and belongs to $\text{SC}(J_{S,v})$ thus we get $A_S = A_{\alpha_0,\ldots,\alpha_3}$ for some coefficients α_i. This implies $a_{i,j} = C_S$ for $i \ne j \in S$ with the constant C_S a priori depending on S. If one interprets the value $a_{i,j}$ as colors of an edge between vertices i and j then we get that all triangles of the complete graph on n vertices are monochromatic. This is possible only if there is only one edge color. So, $a_{i,j} = C$ for $i \ne j$ and one can write $A = A_{\alpha_0,\ldots,\alpha_n}$ with $\alpha_i \in \mathbb{R}$.

Let us find the circumcenter of the empty sphere around $J(n+1,s)$. The point $h_{n+1} = ((\frac{1}{2})^{n+1})$ is at equal distance from all points of $J(n+1,s)$. However, it does not belong to $V_{n,s}$. To find the circumcenter c of $J(n+1,s)$, we take the orthogonal projection of h_{n+1} on the hyperplane $\sum_{i=0}^{n} x_i = s$ for the quadratic form $q_{\alpha_0,\dots,\alpha_n}$. Easy computations give (ii).

Let us prove $\alpha_i > 0$. It is well known that the facets of $J(n+1,s)$ are determined by the inequalities $x_i \geq 0$ and $x_i \leq 1$. It is also easy to see that the Delaunay polytopes adjacent to the facets $x_0 \geq 0$ and $x_0 \leq 1$ are

$$
\begin{aligned}
J_0^- &= \{x \in \{-1,0\} \times \{0,1\}^n \quad : \quad \textstyle\sum_{i=0}^{n} x_i = s\}, \\
\text{and } J_0^+ &= \{x \in \{1,2\} \times \{0,1\}^n \quad : \quad \textstyle\sum_{i=0}^{n} x_i = s\}.
\end{aligned}
$$

The polytopes J_0^-, J_0^+ are equivalent under translation to $J(n+1,s+1)$ and $J(n+1,s-1)$.

The square distance of h_{n+1} to the vertices of $J(n+1,s)$ is $d = \sum_{i=0}^{n} \frac{\alpha_i}{4}$ and the square distance of h_{n+1} to the vertices of J_0^-, J_0^+ not in $J(n+1,s)$ is $d' = \alpha_0 \frac{9}{4} + \sum_{i=1}^{n} \alpha_i \frac{1}{4}$. The conditions defining $SC(J'(n+1,s))$ imply $d' > d$ hence $\alpha_0 > 0$ and by symmetry $\alpha_i > 0$. So, the conditions for $J(n+1,s)$ to be a Delaunay polytope imply that $A = A_{\alpha_0,\dots,\alpha_n}$ with $\alpha_i > 0$. But according to Theorem 1 those conditions are sufficient for the stronger condition of preserving all the Delaunay polytopes of A_n so they are clearly sufficient for just $J(n+1,s)$. $\square$

3. THE POLYTOPES $P(n,s)$

We denote an $(n+1)$-vector whose first a coordinates are A and the remaining $n+1-a$ coordinates B by $(A^a; B^{n+1-a})$. Similar convention is used for vectors with three distinct coordinates, e.g. $(A^a; B^b; C^{n+1-a-b})$.

Definition 1. *Take $n, s \in \mathbb{Z}$ with $s \geq 1$ and $4s \leq n+1$.*
  (i) Set $v_{n,s} = \left(\left(\frac{1}{4}\right)^{4s} ; 0^{n+1-4s} \right)$. The polytope $P(n,s)$ is defined as

$$
P(n,s) = \mathrm{conv} \left\{ v, 2v_{n,s} - v \text{ for } v \in \mathrm{vert}\, J(n+1,s) \right\}.
$$

  (ii) Define $t_{n,s} = \left(\left(\frac{1}{2}\right)^{2s} ; \left(\frac{-1}{2}\right)^{2s} ; 0^{n+1-4s} \right)$ and

$$
V_{n,s}^2 = \{v, t_{n,s} + v \text{ for } v \in V_{n,s}\}.
$$

Theorem 3. *Take $n, s \in \mathbb{Z}$ with $s \geq 2$ and $4s \leq n+1$.*
  (i) $V_{n,s}^2$ is a lattice and $P(n,s)$ affinely generates it.

(ii) The polytope $P(n,s)$ is perfect with the unique, up to positive multiple, positive definite quadratic form being

$$q_{n,s}(x) = 2 \sum_{i=0}^{4s-1} x_i^2 + \sum_{i=4s}^{n} x_i^2.$$

The center of the circumscribed ellipsoid is $v_{n,s}$ and the squared radius is $\frac{3s}{2}$.

Proof. We have $2t_{n,s} \in V_{n,s}$ so $V_{n,s}^2$ is a lattice. $P(n,s)$ generates it since $J(n+1,s)$ generates A_n. By its definition, $P(n,s)$ is centrally symmetric of center $v_{n,s}$. It is well known and easy to prove that if a Delaunay polytope is centrally symmetric then the center c' of its empty sphere coincides with the center c of the antisymmetry operation $v \mapsto 2c - v$. So, we should have $v_{n,s} = c_{\alpha_0,\dots,\alpha_n}$ with

$$c_{\alpha_0,\dots,\alpha_n} = \left(\frac{1}{2} + \frac{C}{\alpha_0}, \dots, \frac{1}{2} + \frac{C}{\alpha_n} \right).$$

Thus:

- For $0 \le i \le 4s-1$, we have $c_i = \frac{1}{4}$. This implies $\alpha_i = -4C$.
- For $4s \le i \le n$, we have $c_i = 0$. This implies $\alpha_i = -2C$.

Summarizing we get $q = -2Cq_{n,s}$ and thus that $P(n,s)$ is perfect. The proof of the Delaunay property follows from the fact that the coefficient in front of x_i^2 are strictly positive for $0 \le i \le n$ and property (i) of Theorem 2. $\qquad\square$

4. The lattice $V_{n,s}^4$

Define the vector $w_{n,s}$ by

$$w_{n,s} = \begin{cases} \left(\left(\tfrac{1}{4}\right)^{2s}, \left(\tfrac{-1}{4}\right)^{2s}, \left(\tfrac{1}{2}\right)^{n+1-4s} \right) - \frac{n+1-4s}{2} e_1 & \text{if } n \text{ is odd,} \\ \left(\left(\tfrac{1}{4}\right)^{2s}, \left(\tfrac{-1}{4}\right)^{2s}, 0, \left(\tfrac{1}{2}\right)^{n-4s} \right) - \frac{n-4s}{2} e_1 & \text{if } n \text{ is even.} \end{cases}$$

Then define

$$V_{n,s}^4 = V_{n,s}^2 \cup w_{n,s} + V_{n,s}^2.$$

Clearly $V_{n,s}^4$ is a lattice that contains $V_{n,s}^2$ as an index 2 sublattice. We want to prove that $P(n,s)$ remains a Delaunay polytope in $V_{n,s}^4$ for some values of n and s.

Theorem 4. *The polytope $P(n,s)$ is a Delaunay polytope of $V_{n,s}^4$ if*

$$6s < \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

Proof. We need to solve the closest vector problem for the lattice $V_{n,s}^4$ and the point $v_{n,s}$. For $V_{n,s}^2$ this is solved by Theorem 3. Thus we need to find the closest vectors in $w_{n,s} + V_{n,s}^2$ to $v_{n,s}$. This is equivalent to finding the closest vectors in $V_{n,s}$ to $v_{n,s} - w_{n,s}$ and to $v_{n,s} - w_{n,s} - t_{n,s}$. We have if n is odd:

$$v_{n,s} - w_{n,s} = \left(0^{2s}; \left(\tfrac{1}{2}\right)^{2s}; \left(-\tfrac{1}{2}\right)^{n+1-4s}\right) + \tfrac{n+1-4s}{2}e_1,$$
$$v_{n,s} - w_{n,s} - t_{n,s} = \left(\left(-\tfrac{1}{2}\right)^{2s}; 1^{2s}; \left(-\tfrac{1}{2}\right)^{n+1-4s}\right) + \tfrac{n+1-4s}{2}e_1,$$

and if n is even:

$$v_{n,s} - w_{n,s} = \left(0^{2s}; \left(\tfrac{1}{2}\right)^{2s}; 0; \left(-\tfrac{1}{2}\right)^{n-4s}\right) + \tfrac{n-4s}{2}e_1,$$
$$v_{n,s} - w_{n,s} - t_{n,s} = \left(\left(-\tfrac{1}{2}\right)^{2s}; 1^{2s}; 0; \left(-\tfrac{1}{2}\right)^{n-4s}\right) + \tfrac{n-4s}{2}e_1.$$

All the vectors occurring have coordinates belonging to $\mathbb{Z}$ or $\mathbb{Z} + \tfrac{1}{2}$. Since the coordinates of elements of $V_{n,s}$ are integral and $q_{n,s}$ has non-zero coefficients only for x_i^2 this gives for $v \in V_{n,s}$ the following lower bounds if n is odd:

$$q_{n,s}(v_{n,s} - w_{n,s} - v) \geq 2 \times 2s \times \tfrac{1}{4} + (n+1-4s)\tfrac{1}{4} = \tfrac{n+1}{4},$$
$$q_{n,s}(v_{n,s} - w_{n,s} - t_{n,s} - v) \geq 2 \times 2s \times \tfrac{1}{4} + (n+1-4s)\tfrac{1}{4} = \tfrac{n+1}{4},$$

and if n is even:

$$q_{n,s}(v_{n,s} - w_{n,s} - v) \geq 2 \times 2s \times \tfrac{1}{4} + (n-4s)\tfrac{1}{4} = \tfrac{n}{4},$$
$$q_{n,s}(v_{n,s} - w_{n,s} - t_{n,s} - v) \geq 2 \times 2s \times \tfrac{1}{4} + (n-4s)\tfrac{1}{4} = \tfrac{n}{4}.$$

So, if n and s satisfy the condition of the theorem then the closest points in $w_{n,s} + V_{n,s}^2$ are at a square distance greater than $\tfrac{3s}{2}$. But $\tfrac{3s}{2}$ is the square radius of the circumscribing sphere thus proving that $P(n,s)$ is a Delaunay polytope in $V_{n,s}^4$. $\square$

The above theorem gives example of Delaunay polytopes, which are perfect but not generating, the first example of which is $P(13,2)$.

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